An Abstract Variational Theorem

Warren B. Moors

Let $(X, \|\cdot\|)$ be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function. Then the **Fenchel conjugate of** f is the function $f^* : X^* \to \mathbb{R} \cup \{\infty\}$ defined by,

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in X\}.$$

In this talk we will present a proof of the following theorem:

Theorem: Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function on a Banach space $(X, \|\cdot\|)$. If there exists a nonempty open subset A of $\mathrm{Dom}(f^*)$ such that $\mathrm{argmax}(x^* - f) \neq \emptyset$ for each $x^* \in A$, then there exists a dense and G_δ subset R' of A such that $(x^* - f) : X \to \mathbb{R} \cup \{\infty\}$ has a strong maximum for each $x^* \in R'$. In addition, if $0 \in A$ and $\varepsilon > 0$ then there exists an $x^* \in X^*$ with $\|x^*\| < \varepsilon$ such that $(x^* - f) : X \to \mathbb{R} \cup \{\infty\}$ has a strong maximum.

Some applications of this theorem will also be presented, [1, 2, 3, 4, 5, 6].

References

[1] R. Baire, *Sur les fonctions de variables réelles*, Annali di Mat. (3) **3** (1899), 1–123

[2] J. Borwein and D. Preiss, A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Trans. Amer. Math. Soc.* **303** (1987), no. 2, 517–527.

[3] R. Deville, G. Godefroy and V. Zizler, A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions. *J. Funct. Anal.* **111** (1993), no. 1, 197–212.

[4] I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.

[5] R. C. James, Weakly compact sets. *Trans. Amer. Math. Soc.* **113** (1964), 129–140.

[6] W. B. Moors, A gentle introduction to James' weak compactness theorem and beyond, submitted to *Methods Funct. Anal. Topology.*